Trigonometric Identities

Euler's Formula

$$e^{iy} = \cos y + i\sin y$$

Proof

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots$$ Taylor expansion of e^x

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(\frac{y}{1!} - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)$$ Since we can rearrange absolutely convergent sums

$$= \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} y^{2n}}{(2n)!}\right) + i\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} y^{2n-1}}{(2n-1)!}\right)$$

$= \cos y + i\sin y$. Q.E.D. Taylor expansion of $\cos y$ and $\sin y$

Identities (Addition under sin and cos)

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

Proof

$$e^{ix}e^{iy} = e^{i(x+y)} \Rightarrow (\cos x + i\sin x)(\cos y + i\sin y) = \cos(x+y) + i\sin(x+y)$$
$$\Rightarrow \cos x\cos y + i\sin y\cos x + i\sin x\cos y - \sin x\sin y = \cos(x+y) + i\sin(x+y)$$
$$\Rightarrow (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \sin y\cos x) = \cos(x+y) + i\sin(x+y)$$

By matching up the coefficients, we have:

$$\sin(x+y) = \sin x\cos y + \sin y\cos x$$
$\cos(x+y) = \cos x\cos y - \sin x\sin y$. Q.E.D.

Identities (Subtraction under sin and cos)

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Proof

$$e^{ix}e^{-iy} = e^{i(x-y)} \Rightarrow (\cos x + i\sin x)(\cos y - i\sin y) = \cos(x-y) + i\sin(x-y)$$
$$\Rightarrow \cos x\cos y - i\sin y\cos x + i\sin x\cos y + \sin x\sin y = \cos(x-y) + i\sin(x-y)$$
$$\Rightarrow (\cos x\cos y + \sin x\sin y) + i(\sin x\cos y - \sin y\cos x) = \cos(x-y) + i\sin(x-y)$$

By matching up the coefficients, we have:

$$\sin(x-y) = \sin x\cos y - \sin y\cos x$$
$\cos(x-y) = \cos x\cos y + \sin x\sin y$. Q.E.D.

Identities (Double Angle Formulae)

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

Proof

Using the addition identities above, (with $x = y$), we have:

$$\sin 2x = \sin x \cos x + \cos x \sin x = 2\sin x \cos x$$
$$\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$$ Q.E.D.

Identity (Sum of squares)

$$\sin^2 x + \cos^2 x = 1$$

Proof

$$e^{ix}e^{-ix} = e^0 = 1$$
$$\Rightarrow (\cos x + i\sin x)(\cos x - i\sin x) = 1$$
$$\Rightarrow \cos x \cos x - i\sin x \cos x + i\sin x \cos x + \sin x \sin x = 1$$
$$\Rightarrow \cos^2 x + \sin^2 x = 1.$$ Q.E.D.

Identities (Pythagorean Identities)

$$\tan^2 x + 1 = \sin^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Proof

We know that $\sin^2 x + \cos^2 x = 1$.

Dividing both sides by $\cos^2 x$, we have:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

If we instead divide both sides by $\sin^2 x$, we have:

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x.$$ Q.E.D.

Identities (Half-Angle formulae)

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Proof

Recall that $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin^2 x + \cos^2 x = 1$. Solving the second identity for $\cos^2 x$ and substituting it into the first identity, we have:

$$\cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x \Rightarrow 2\sin^2 x = 1 - \cos 2x$$
$$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

If we instead solve the second identity for $\sin^2 x$ and substitute it into the first identity, we have:

$$\cos 2x = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = \cos 2x + 1$$

$$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}.$$ Q.E.D.

Identities (Product Identities)

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

Proof

Recall the identities $\cos(x-y) = \cos x \cos y + \sin x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$. If we subtract the second from the first:

$$\cos(x-y) - \cos(x+y) = \sin x \sin y + \sin x \sin y = 2\sin x \sin y$$

$$\Rightarrow \sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

If we add the two identities instead, we have:

$$\cos(x-y) + \cos(x+y) = \cos x \cos y + \cos x \cos y = 2\cos x \cos y$$

$$\Rightarrow \cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

Recall the identities $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\sin(x-y) = \sin x \cos y - \cos x \sin y$. If we add the two identities, we have:

$$\sin(x+y) + \sin(x-y) = \sin x \cos y + \sin x \cos y = 2\sin x \cos y$$

$$\Rightarrow \sin x \cos y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

If instead, we subtract the second identity from the first, we have:

$$\sin(x+y) - \sin(x-y) = \cos x \sin y + \cos x \sin y = 2\cos x \sin y$$

$$\Rightarrow \cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right).$$ Q.E.D.